EFFECTS OF NOISE ON SIGNAL RECONSTRUCTION FROM FOURIER TRANSFORM PHASE *

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ABSTRACT

The effects of noise in the given phase on signal reconstruction from the Fourier transform phase are studied. Specifically, the effects of different methods of sampling the degraded phase, of the number of non-zero points in the sequence, and of the noise level on the sequence reconstruction are examined. A sampling method is developed to significantly reduce the error in the reconstructed sequence, and the error is found to increase as the number of non-zero points in the sequence increases and as the noise level increases. In addition, an averaging technique is developed which reduces the effects of noise when the continuous phase function is known. Finally, as an illustration of how the results in this paper may be applied in practice, Fourier transform signal coding is considered. Coding only the Fourier transform phase and reconstructing the signal from the coded phase is found to be considerably less efficient (i.e. a higher bit rate is required for the same mean square error) than reconstructing from both the coded phase and magnitude.

I. INTRODUCTION

Reconstruction of a discrete time signal or sequence from its Fourier transform phase has a variety of potential applications. For example, in phase-only holograms, known as "kinoforms" (1), the Fourier transform magnitude information is lost while the phase is retained. If the magnitude information, thus the signal, could be recovered from the phase information alone, the quality of images reconstructed from kinoforms could be significantly improved.

Although a sequence is not in general recoverable from the phase information alone, under certain conditions which are satisfied in many practical cases of interest, a sequence can be reconstructed from the phase information alone. Specifically, Hayes, Lim, and Oppenheim (2) have recently shown that a finite duration sequence, provided its z-transform has no zeroes in reciprocal pairs or on the unit circle, is uniquely specified to within a scale factor by its Fourier transform phase.

Even though the results by Hayes, Lim, and Oppenheim (2) have important theoretical significance, they are limited in practice since they are based on the assumption that the exact phase is available. In many potential application problems, the available phase may have been degraded by measurement noise, quantization noise, etc. To understand the effects of phase degradation on the reconstructed sequence, a series of experiments has been performed. In this paper, we present the experimental results and propose a technique that reduces the phase degradation effects when the continuous phase function is available.

The organization of this paper is as follows. In Section II, important theoretical results relevant to this paper are summarized. A discussion of the phase-only signal reconstruction algorithm used in the experiments is also given. In Section III, a series of experiments that have been performed is discussed and the results are presented. In Section IV, we illustrate how the results in Section III may be applied in practice. In Section V, a technique to reduce the effects of phase degradation is discussed. Finally, a summary of the major results of this paper is presented in Section VI.

II. SUMMARY OF PREVIOUS THEORETICAL RESULTS

Let x(n) and y(n) be two finite length sequences whose z-transforms have no zeroes in reciprocal pairs or on the unit circle. Let $\theta_x(\omega)$ and $\theta_y(\omega)$ denote the Fourier transform phases of x(n) and y(n) respectively. It can be shown (2) that if $\theta_x(\omega) = \theta_y(\omega)$ at (N - 1) distinct frequencies between zero and π, then x(n) = Cy(n) for some positive constant C. Moreover, if $\tan \theta_x(\omega) = \tan \theta_y(\omega)$ at (N - 1) distinct frequencies between zero and π, then x(n) = Cy(n) for some real constant C.

To reconstruct the sequence that satisfies the above conditions from its Fourier transform phase samples, two numerical algorithms have been developed. The first is an iterative algorithm which improves the estimate in each iteration. The second is a non-iterative algorithm which reconstructs the sequence by solving a set of linear equations. In this paper, the non-iterative algorithm has been used exclusively since it leads to the desired solution without any iterations and is very flexible in choosing the frequencies at which the phase function is sampled.

The non-iterative algorithm can be derived (2) from the definition of the Fourier transform phase. Specifically, by expressing $\tan \theta_x(\omega)$ as

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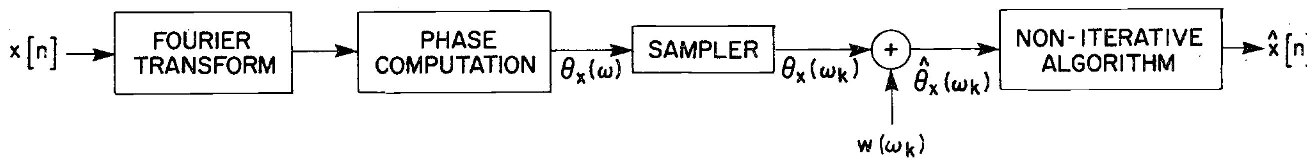


Figure 1.--Experiments performed to study the effect of noise on signal reconstruction from Fourier transform phase.

the imaginary part of the Fourier transform divided by the real part and by some algebraic manipulations, it can be shown (2) that

$$\sum_{n=1}^{N-1} x(n) \sin [\theta_x(\omega) + n\omega] = -x(0) \sin \theta_x(\omega) \qquad (1)$$

By sampling $\theta_x(\omega)$ at (N - 1) frequencies between zero and π, Equation (1) can be expressed in matrix form as

$$S\underline{x} = -x(0)\underline{b} \qquad (2)$$

where $\underline{x}$ is a column vector containing the values of x(n) for $1 \leq n \leq N - 1$ and x(0) is the unknown scaling factor. The matrix S in Equation (2) can be shown to have an inverse and the vector $\underline{x}$ can be determined from

$$\underline{x} = -x(0)S^{-1}\underline{b} \qquad (3)$$

From Equation (3), the major computation involved in the non-iterative algorithm is the inversion of an (N - 1) × (N - 1) matrix which, as N gets large, becomes more difficult and may give rise to severe round-off errors resulting in numerical instability. This potential problem has been avoided by limiting the experiments to relatively small values of N and by detecting (3) the occurrence of numerical instability in each reconstructed sequence.

The above results have also been extended (4) to two-dimensional signals.

III. EXPERIMENTS

The reconstruction process used in this study is illustrated schematically in Figure 1. In considering this process, several points are noted.

First, the input data, x(n), denotes a sequence which satisfies the conditions given in Section II. The input sequences were generated digitally from a zero-mean white Gaussian random process.

Second, digitally generated white noise was added directly to the undegraded phase to obtain the degraded phase. Each noise sample is statistically independent of all other noise samples, and was obtained from a uniform probability density function given by

$$P_w(w_0) = \begin{cases} \frac{1}{2w_\ell} & -w_\ell < w_0 < w_\ell \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

where w_ℓ denotes the noise level. The noise levels of interest lie in the range $\pi \times 10^{-5} \leq w_\ell \leq \pi \times 10^{-1}$ since, for most sequences, noise below $\pi \times 10^{-5}$ had negligible effects upon the reconstructed sequence, whereas noise above $\pi \times 10^{-1}$ had severe effects.

Third, to quantify the effects of phase degradation on the reconstructed sequence, the normalized mean-square error (NMSE) was computed from

$$\text{NMSE} = \frac{\sum_{n=0}^{N-1} (x(n) - k\hat{x}(n))^2}{\sum_{n=0}^{N-1} x^2(n)} \qquad (5)$$

with k chosen to minimize the NMSE.

Finally, to determine the effects of a particular experimental parameter on the reconstruction error, the reconstruction of Figure 1 was implemented for 1000 sequences. From the resulting reconstructed sequences, the mean of the NMSE was computed. The mean of the log of the NMSE (LOGNMSE) was also computed to detect those cases in which the average NMSE computed is primarily due to very large errors in a small fraction of the 1000 sequences.

The effects of phase degradation were examined first as a function of the sampling method. If the exact phase is available, the NMSE is zero independent of at what frequencies the N - 1 phase samples are obtained. When the phase is degraded, however, the NMSE depends on the specific sampling method.

To determine the sampling method that leads to the smallest average NMSE, a number of different sampling strategies have been considered. Among these different methods, choosing N - 1 frequencies (ω_i for $1 \leq i \leq N - 1$) such that $\omega_1 = \pi/2(N - 1)$ and $\Delta\omega = \omega_i - \omega_{i-1} = \pi/(N - 1)$ for $2 \leq i \leq N - 1$ has been observed to lead to the smallest average NMSE. This choice of frequencies minimizes the maximum separation between two consecutive frequencies under the interpretation that $\omega = 0$ and $\omega = \pi$ are connected. In addition, the frequencies chosen are symmetric with respect to $\omega = \pi/2$. Examples of this choice of N - 1 frequencies are shown in Figure 2 for N = 5 and 8.

With the (N - 1) samples of $\hat{\theta}_x(\omega)$ obtained at frequencies ω_i with $\omega_1 = \pi/2(N - 1)$ and $\Delta\omega = \pi/(N - 1)$, the effects of the sequence length N and noise level w_ℓ on the reconstructed sequence were considered. The values of N and w_ℓ used are N = 4, 8, 16, 32, 64, and 128 and $w_\ell = \pi \times 10^{-1}$, $\pi \times 10^{-2}$, $\pi \times 10^{-3}$, $\pi \times 10^{-4}$, and $\pi \times 10^{-5}$. The average NMSE and LOGNMSE for these values of N and w_ℓ are

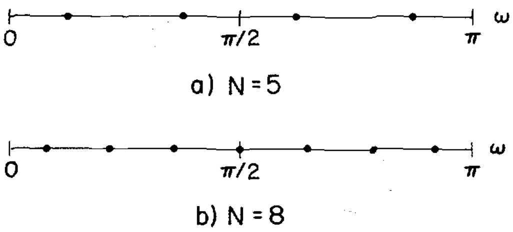
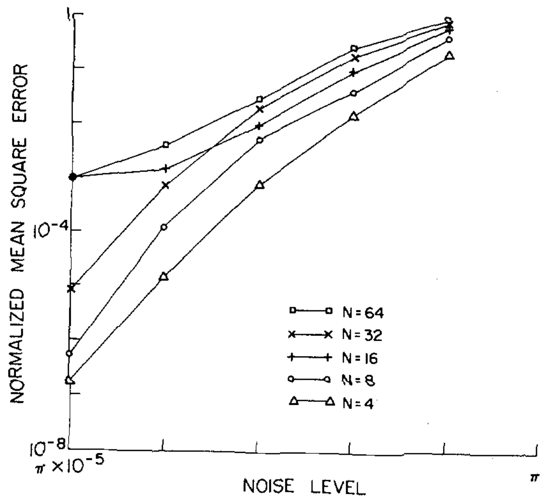
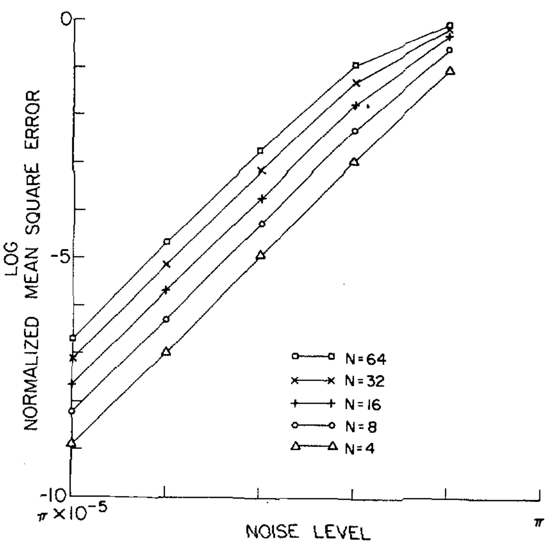


Figure 2.--Examples of N - 1 frequencies chosen with $\Delta\omega = \pi/(N - 1)$ and $\omega_1 = \pi/2(N - 1)$



(a)



(b)

Figure 3.--(a) Normalized mean square error as a function of data length N and noise level w_ℓ;

$$\Delta\omega = \frac{\pi}{(N - 1)}, \quad \omega_1 = \frac{\pi}{2(N - 1)}$$

(b) Log normalized mean square error as a function of data length N and noise level w_ℓ;

$$\Delta\omega = \frac{\pi}{(N - 1)}, \quad \omega_1 = \frac{\pi}{2(N - 1)}$$

shown in Figure 3. The results in Figure 3 show that the average NMSE and LOGNMSE increase as the noise level increases and the sequence length increases.

IV. APPLICATIONS

The results in Section III may be useful in some practical situations in which phase-only signal reconstruction is considered. In this section, we illustrate one such example.

In Fourier transform image coding, both the phase and the magnitude are coded and an image is reconstructed from the coded phase and magnitude. For monochrome images, the magnitude and phase may be coded at bit rates of 1.0 to 1.5 bits/pixel with mean square error distortion less than 0.5% (5). Since an image can be reconstructed from its Fourier transform phase alone, we may consider coding only the phase and then using the phase-only signal reconstruction algorithm to reconstruct the signal from the coded phase. Assuming that the phase is quantized by a uniform quantizer, the bit rate required to achieve the quantization noise level w_ℓ is given by Reference (3):

$$B = \log_2(\pi/w_\ell) \qquad (6)$$

where B represents the number of bits in each codeword. From Figure 3, to achieve the average NMSE of 1% for N = 64 (this corresponds to a subimage size of 8 x 8 pixels), the noise level w_ℓ should be less than $\pi \times 10^{-3}$ and therefore, from Equation (6), requires more than 10 bits/pixel. Even though the NMSE is not exactly the same as the mean square error used in image-coding literature, the above results show that both a low distortion rate and a low bit rate cannot be achieved by attempting to code only the Fourier transform phase and then reconstructing the image from the coded phase using the phase-only signal reconstruction algorithm.

In addition to the Fourier transform image-coding problem, the results in Section III may be useful in other applications, such as speech enhancement, where one may consider first estimating the phase more accurately from the degraded speech, and then attempting to reconstruct the signal from the estimated phase information.

V. SIGNAL RECONSTRUCTION FROM MORE THAN N - 1 PHASE SAMPLES

If more than N - 1 phase samples are available for signal reconstruction, then the additional information may be used to reduce the signal reconstruction error. One approach we have considered to exploit the additional information is to average several reconstructed sequences obtained from different sets of N - 1 phase samples. That is, if $\hat{x}_1(n)$ is obtained from one set of N - 1 phase samples and $\hat{x}_2(n)$ is obtained from a different set of N - 1 phase samples, then $\hat{x}(n) = (\hat{x}_1(n) + \hat{x}_2(n))/2$ may give a better estimate of x(n) than either $\hat{x}_1(n)$ or $\hat{x}_2(n)$.

To test if averaging the reconstructed sequences reduces the error, experiments were

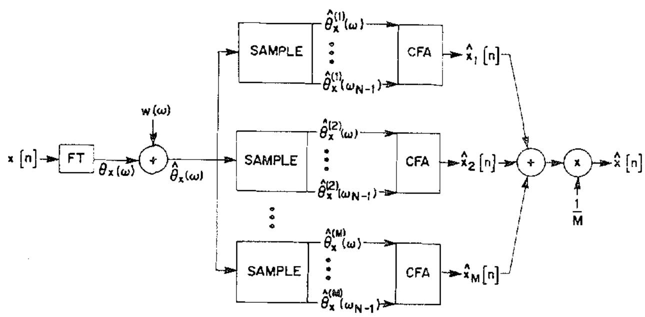
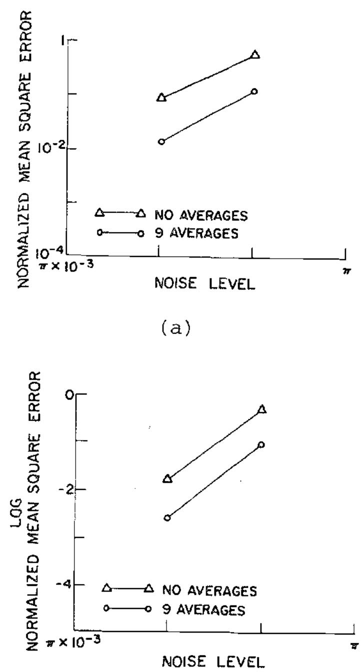


Figure 4.--Experiments performed to study the effect of averaging more than one reconstructed sequence.

performed using the averaging process depicted in Figure 4 with $\Delta\omega = \pi/(N - 1)$. The results are shown in Figure 5. It is clear from the figure that averaging reduces the NMSE in the reconstructed sequences. Additional experiments showed that as the number of reconstructed sequences used in the averaging increases, the average NMSE decreases, but at a lower rate.



(a)

(b)

Figure 5.--(a) Performance improvement in NMSE by averaging.

$$N = 16 \quad , \quad \Delta\omega = \frac{\pi}{(N - 1)}$$

Values of ω_1 used are:

$$\omega_1 = \frac{4\pi}{16(N - 1)}, \frac{5\pi}{16(N - 1)}, \ldots \frac{12\pi}{16(N - 1)}$$

(b) Performance improvement in LOGNMSE by averaging.

$$N = 16 \quad , \quad \Delta\omega = \frac{\pi}{(N - 1)}$$

Values of ω_1 used are:

$$\omega_1 = \frac{4\pi}{16(N - 1)}, \frac{5\pi}{16(N - 1)}, \ldots \frac{12\pi}{16(N - 1)}$$

VI. CONCLUSION

In this paper, we have studied the effect of phase degradation on the signal reconstruction error, using the non-iterative signal reconstruction algorithm. A number of different sampling methods have been considered and the sampling method that appears to minimize the average NMSE has been determined. Using this sampling method, the average NMSE and average LOGNMSE were computed as a function of the sequence length and the noise level.

The usefulness of phase-only reconstruction in Fourier transform image coding was then considered as an example that illustrates how the results of this paper may be used in practice. Our analysis shows that reconstructing an image from the coded phase using the phase-only signal reconstruction algorithm is considerably less efficient in the bit rate than reconstructing the image from the coded phase and magnitude.

Finally, to reduce the effects of phase degradation, an averaging technique was developed which reconstructs the signal from more than (N - 1) phase samples. This technique can significantly reduce the error and may be used in those applications in which continuous phase is available.

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